Minimum Viable Example of the Transition Matrix Problem

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In this set of notes I present a minimum viable example of the transition matrix problem. On the one hand it is so simple as to be trivial to solve by hand on the other it's just hard enough not to be obvious. The reason for writing this set of notes is that I have had trouble explaining to people exactly what problem I am solving in my maze generation notes or why I think having an algorithmic means of solving this problem at scale is non-trivial. I have chosen to call this the transition matrix problem though it may have other names and below is a minimum viable example of that problem.

The Ball Game

Consider the following selection game. You must select a sequence of balls from a set of 4 types with the following conditions.

$$b_1 \to \{b_2, b_3\} \tag{1}$$
$$b_2 \to \{b_1, b_3\} \tag{2}$$
$$b_3 \to \{b_2, b_4\} \tag{3}$$
$$b_4 \to \{b_2, b_3\} \tag{4}$$

where condition $b_i \to \{b_j, b_k\}$ is a rule stating that if you previously selected b_i your next selection must be b_j or b_k.

You may choose your initial ball selection as you wish. After you have selected n balls, your selection will be evaluated on two criteria. The first is the **ball number criteria** which states, the number of each ball type selected must be approximately $\frac{n}{4}$. The second is the **randomness criteria** which states, the order in which the types of balls are selected must be random for any subsequence longer then 2. If your selection passes both criteria you win. How would you proceed in selecting balls in order to win this game?

The Solution

Fig. 1 represents The Ball Game as Markov chain where each state represents a selection of a ball with the corresponding number and all possible transitions are represented by directed arrows. This representation makes it clear that at least one solution to the problem is to select transition probabilities such that the sequence of selected balls satisfy the first and second criteria.

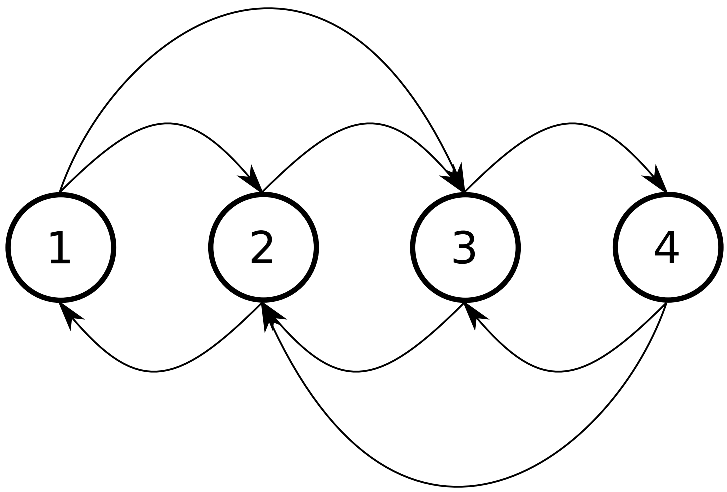

Figure 1: Representation of The Ball Game

To see that not all assignments of transition probabilities will satisfy the ball number and randomness criteria try assigning equal transition probability to all edges leaving a state. This produces Fig. 2.

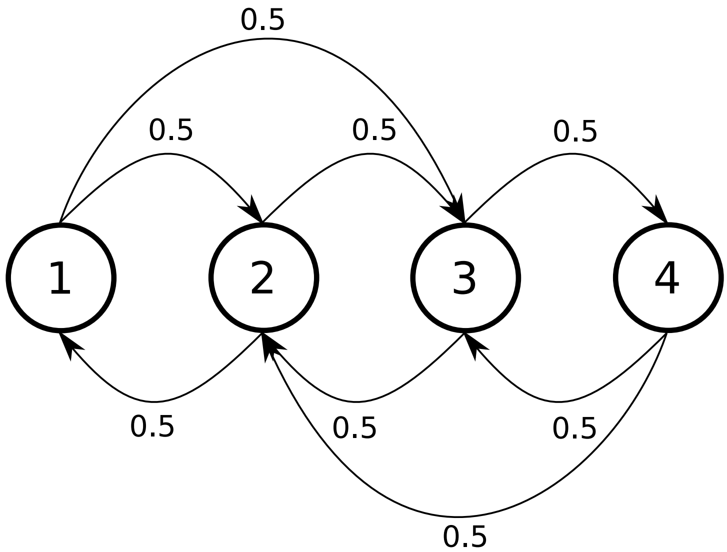

Figure 2: Proposed Solution

Clearly starting in any state of the Markov chain in Fig. 2 will produce a random sequence and pass the randomness criteria. But turns out that it does not meet the ball number criteria. To see this we solve for the steady state probabilities of this Markov chain.

Let q_i be the steady state probabilities in the Markov chain and $p = \frac{1}{2}$. The steady state probability equations for the Markov chain are then,

$$q_1 = q_2 p \tag{5}$$
$$q_2 = p(q_1 + q_3 + q_4) \tag{6}$$
$$q_3 = p(q_1 + q_2 + q_4) \tag{7}$$
$$q_4 = q_3 p \tag{8}$$
$$1 = q_1 + q_2 + q_3 + q_4 \tag{9}$$

From equation (9) it follows that,

$$q_1 + q_3 + q_4 = 1 - q_2 \tag{10}$$
$$q_1 + q_2 + q_4 = 1 - q_3 \tag{11}$$

Substitute equations (10) into (6) gives,

$$
\begin{aligned}
q_2 &= p(1 - q_2) && (12)\\
q_2 &= p - pq_2 && (13)\\
q_2(1 + p) &= p && (14)\\
q_2 &= \frac{p}{1 + p} && (15)\\
q_2 &= \frac{1}{3} && (16)
\end{aligned}
$$

By symmetry it follows that,

$$q_3 = \frac{p}{1 + p} = \frac{1}{3} \tag{17}$$

Finally given the steady state equations (5) and (8) it follows that

$$q_1 = \frac{p^2}{1 + p} = \frac{1}{6} \tag{18}$$

$$q_4 = \frac{p^2}{1 + p} = \frac{1}{6} \tag{19}$$

As not all steady state probabilities q_i are equal, clearly the Markov chain in Fig. 2 fails to satisfy the number of balls criteria.

How should we assign transition probabilities to solve The Ball Problem? By inspection it quickly becomes clear that the following transition matrix solves The Ball Problem,

$$\mathbf{T} = \begin{bmatrix} 0, & \frac{1}{2}, & \frac{1}{2}, & 0 \\ 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1 \\ 0, & \frac{1}{2}, & \frac{1}{2}, & 0 \end{bmatrix} \tag{20}$$

which can be represented as the Markov chain show in Fig. 3.

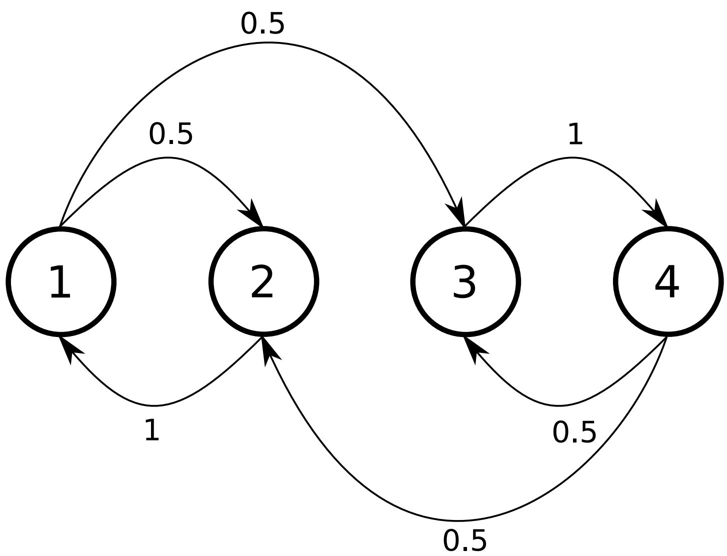

Figure 3: The Ball Game Solution

The steady state equations this Markov chain are,

$$q_1 = q_2 \tag{21}$$
$$q_2 = p(q_1 + q_4) \tag{22}$$
$$q_3 = p(q_1 + q_4) \tag{23}$$
$$q_4 = q_3 \tag{24}$$
$$1 = q_1 + q_2 + q_3 + q_4 \tag{25}$$

Equating (22) and (23) gives,

$$q_2 = q_3 \tag{26}$$

Equating (21) and (26) gives,

$$q_1 = q_3 \tag{27}$$

Substituting (24), (26), and (27) into (25) gives,

$$1 = q_3 + q_3 + q_3 + q_3 \tag{28}$$
$$q_3 = \frac{1}{4} \tag{29}$$

Therefore we have $q_1 = q_2 = q_3 = q_4 = \frac{1}{4}$ as desired.

However clearly solving instances of the Transition Matrix Problem such as The Ball Problem where you are given a set of target steady state probabilities and possible transition edges and asked to assign transition probabilities to these edges such that the target steady state probabilities are met cannot generally be accomplished by inspection. This is the value of an algorithm which assigns these transition probabilities, generating a transition matrix which realizes target steady state probabilities.